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INTRODUCTION

After each census, many of the states comprising the United States undertake a voter redistricting process. A state may need to undertake this process because of losing or gaining seats from the reapportionment process or due to population shifts within the state. Districting occurs at many levels of government. At the federal level, a state creates districts for the House of Representatives. Many states also create districts for the state house and senate. The goal of this process is to implement the constitutional mandate of equal representation.

At the congressional level, the process, rules, and central issues vary from state to state but all voter redistricting plans must satisfy two legal requirements: 1) The districts must have equal populations. 2) The districts must be *contiguous*. A district is contiguous if one can go from any point in the district to any other point in the district without leaving the district (Garfinkel and Nemhauser, 1970). In addition, districts are often required to be *compact*. A district is geographically compact if the shape of the district is approximately square or circular. A district has population compactness if the population is clustered close together.

Strict adherence to equal-population districts has resulted in an increasing number of congressional-level plans that have a maximum deviation of less than 1%. The driver for strict adherence to district population equality, Article I, section 2 of the Constitution, does not cover state legislatures. Consequently, a looser standard for population equality has been applied. "The courts seemed to point toward an informal 10% guideline [Gaffney v. Cummings 412 U.S. 735 (1973)]: The Court would tolerate deviations up to plus or minus 5% of the ideal population in order for a state legislative district to accommodate other valid reapportionment goals." (Butler and Cain, p. 31, 1992).

The state constitution of Kentucky mandates that the State will be divided into 38 Senatorial Districts and 100 Representative Districts for state government. These districts "must be as nearly equal in population as may be without dividing any county, except where a county may include more [population] than one district" (Kentucky Constitution, Section 33). In addition the counties that form a district shall be contiguous. In 1994, the Kentucky Supreme Court invalidated the 1991-redistricting plan in *Fischer v. State Board of Elections* because too many counties were divided. There existed other plans that had fewer counties divided. The court required a redistricting plan to have the minimum number of county divisions and satisfy the equal population requirement. However, this minimum was not known (Kearns, 1995).

We will focus on the Kentucky redistricting problem with a goal of minimizing the number of county divisions while maintaining these constraints:1) The districts have equal populations (±5%) 2) The districts are contiguous.

In past work, Camm (1995) has formulated a model for the minimum-cut redistricting problem. The objective of this model is to determine the minimum number of county cuts such that population equality and contiguity are satisfied. In 1995, Lindo, an optimization software package did not find a feasible solution. In 1996, we unsuccessfully tried to use CPLEX to solve the Kentucky problem. This convinced us of the need for additional work on this problem.

This is the first step in further understanding the minimum-cut redistricting problem. We utilize Camm's model and CPLEX to solve several redistricting problems. This will also create a baseline for comparing other models and solution techniques. In section 2 we look at a sampling of other work on districting problems. In section 3, we introduce the 1-ring and 2-ring mixed integer programming models. In section 4, we describe our computation experience with these models. Finally in section 5, we describe our conclusions.

OTHER SOLUTION APPROACHES

There is 30 years of work on various districting and redistricting problems in the operations research literature. Hess et al. (1965) develop an IP formulation analogous to the warehouse-location problem. Using a heuristic approach, they build legislative districts for the state of Delaware from census enumeration districts.

Garfinkel and Nemhauser (1970) developed an algorithm to find all optimal solutions for a version of the redistricting problem. The goal of this version is to minimize the maximum deviation of the district population from the desired population. The problem was constrained so that the districts would be contiguous, compact and so that the district population could not exceed a specified percentage. A two-phase solution process was developed that was successfully applied to states with less than 40 counties. Attempts to solve a 55 county state were unsuccessful.

Plane (1982) criticizes the use of the warehouse-location model for redistricting. Specifically, he criticizes the use of a population centroid measure for the objective function as inappropriate. He proposes a measure of aggregate lengths of "interpersonal separations" as a more appropriate goal for achieving "communities of interest." This leads to a quadratic integer program formulation for the redistricting problem.

Mehrotra, Johnson, and Nemhauser (1998) study a version of the districting problem that minimizes the sum of the compactness of all districts in the plan. Again, their model ensures population equality and district contiguity. They have applied their solution procedure to the states of South Carolina and North Carolina.

Birge (1983) studied a problem that is closer to the Kentucky redistricting problem than that of Garfinkel and Nemhauser (1970) and Mehrotra, Johnson, and Nemhauser (1998). The objective is to minimize the number of existing political units that do not belong to a single district. The constraints for this formulation are that the population of each district must be within specified bounds and the districts must be contiguous. A heuristic procedure was developed and used for a problem from the state of Michigan.

The creation of districts from smaller population units is not only applicable to voter redistricting. Carey, Srinivasan, and Strauss (1996) have examined the consolidation of municipalities in order to reduce the costs of providing public services. Over the last 30 years, much work has also been done on the school district planning problem which assigns students to a school district in order to optimize some criteria such as travel distance. Most recently, Lemsberg and Church (2000) have formulated the school boundary stability problem over time which minimizes the impact of change on the students while including the traditional distance objective to encourage compact boundaries.

MINIMUM-CUT REDISTRICTING MODEL

We define the following notation that will be used throughout the paper.

K – number of districts
NC – number of counties
T - target district population = (state population)/ *K*

We will constrain the population of a district to be within ±5% of the target district population.

We present two variations of this model: single-ring and two-ring. Both variations select *K* counties to be district seats. The remaining counties are assigned to one of these district seats. Counties that are larger than the upper population limit for a district must wholly contain at least one district and therefore require a cut.

Single-ring Adjacency Model

In the single-ring adjacency model, only counties immediately adjacent to the district seat may be assigned to that district seat. See Figure 1.

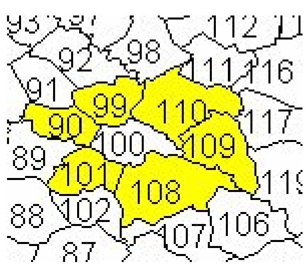

Figure 1 Single-ring adjacency for county 100

The following data are defined:

L: lower population bound for any district, 0.95*T

U: lower population bound for any district, 1.05*T

Pop_i: total population of county i

M_i: Min$\{Pop_i, U\}$

$MinDistricts_i$: Minimum number of counties wholly contained by large county i;

$MinDistricts_i = \lfloor Pop_i/U \rfloor$

$MinFrac$: Minimum percentage of population that must be assigned

The following sets are defined:

C: Set of all counties

B: Counties such that $Pop[i] \geq U$; Counties wholly contain at least one district

O: Set of counties that do not wholly contain any district = $C \setminus L$

N_j: Counties that are immediately adjacent to county j

Decision Variables:

y_j $\begin{cases} 1 & \text{if county } j \text{ is a district "seat"} \\ 0 & \text{otherwise} \end{cases}$

$j \in C$

x_{ij} $\begin{cases} 1 & \text{if part of county } i \text{ is assigned to the district with } j \text{ as seat} \\ 0 & \text{otherwise} \end{cases}$

$i \in N_j;\ j \in C$

p_{ij} population of county i assigned to district with j as seat

$i \in N_j;\ j \in C$

c_i number of cuts made to county i

$i \in C$

z_i Number of districts wholly contained in county i

$i \in B$

pz_i Population of the districts wholly contained in county i

$i \in B$

Model:

MIN

$$\sum_{i \in C} C_i \tag{1}$$

Subject to:

$$x_{ij} \leq y_j \qquad i \in N_j; j \in C \tag{2}$$

$$\sum_{j \in B} z_j + \sum_{j \in C} y_j = K \tag{3}$$

$$Ly_j \le \sum_{i \in N_j} p_{ij} \le Uy_j \qquad j \in O \tag{4}$$

$$Lz_j \le pz_j \le Uz_j \qquad j \in B \tag{5}$$

$$\sum_{j \in N_i} p_{ij} = Pop_i \qquad i \in O \tag{6}$$

$$pz_i + \sum_{j \in N_i} p_{ij} = Pop_i \qquad i \in B \tag{7}$$

$$p_{ij} \le M_i x_{ij} \qquad i \in C;\ j \in N_i \tag{8}$$

$$p_{ij} \ge [MinFrac(Pop_i)]x_{ij} \qquad i \in C;\ j \in N_i \tag{9}$$

$$c_i = \sum_{j \in N_i} x_{ij} - 1 \qquad i \in O \tag{10}$$

$$c_i = \sum_{j \in N_i} x_{ij} + z_i - 1 \qquad i \in B \tag{11}$$

$$x_{ii} \ge y_i \qquad i \in C \tag{12}$$

$$p_{ii} \ge M_i y_i \qquad i \in C \tag{13}$$

$$z_i \ge MinDistricts_i \qquad i \in C \tag{14}$$

The objective (1) is to minimize the number of cuts that are made to counties. Constraint (2) requires that a county cannot be assigned to a district seat if the seat has not been selected. Constraint (3) sets the number of districts to be created. Constraints (4) and (5) require that the population for each district be within the designated limits. Constraint 5 constrains counties with a large population ($Pop_i \ge U$). Constraints (6) and (7) require that a county's entire population be assigned. Constraint (8) limits the population that can be assigned to district *j* to the minimum of the population in the district and the district upper population limit. Constraint (9) requires that if population is assigned from county *i* to seat *j* then at least *MinFrac*% of the population is assigned. Constraints (10) and (11) count the number of cuts. Constraint (12) and (13) require that if a county is a district seat then all of the county's population is assigned to this seat. Constraint (14) states that the minimum number of county cuts for a large county is *MinDistrict*.

Two-Ring Adjacency Model

We extend the Single-ring Adjacency Model to allow counties that are not immediately adjacent to a district seat to be in that district. The immediately adjacent counties form a single ring around the district seat. We add a second ring of counties that are immediately adjacent to the counties in the single ring (See Figure 2).

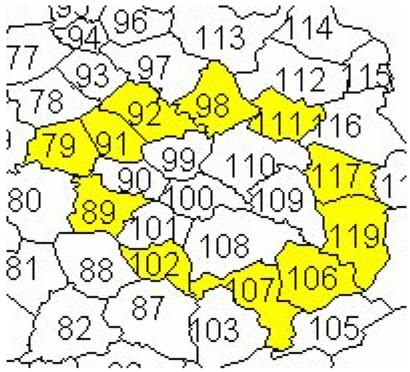

Figure 2 Two-ring adjacency for County 100

We define an additional set:

R_j Counties in the second adjacency ring of county j

We introduce two more sets of variables

r_{ij} $\begin{cases} 1 & \text{if part of county } i \text{ is assigned to the district with } j \text{ as seat} \\ 0 & \text{otherwise} \end{cases}$

$i \in R_j;\ j \in C$

q_{ij} population of county i assigned to district with j as seat

$i \in R_j;\ j \in C$

This model requires these additional constraints:

$$r_{ij} \le x_{tj} \qquad i \in R_j; j \in C; t \in N_j \qquad (15)$$

$$q_{ij} \le M_i r_{ij} \qquad i \in C; j \in R_i \qquad (16)$$

$$q_{ij} \le M_i r_{ij} \qquad i \in C;\ j \in N_i \qquad (17)$$

$$q_{ij} \ge [MinFrac(Pop_i)] r_{ij} \qquad i \in C;\ j \in N_i \qquad (18)$$

Constraints (4), (6) and (7), (10) and (11) are modified to include two ring variables.

$$Ly_j \le \sum_{i \in N_j} p_{ij} + \sum_{i \in R_{ij}} q_{ij} \le Uy_j \qquad j \in C \qquad (19)$$

$$\sum_{j \in N_i} p_{ij} + \sum_{j \in R_{ij}} q_{ij} = Pop_i \qquad i \in O \qquad (20)$$

$$pz_i + \sum_{j \in N_i} p_{ij} + \sum_{j \in R_{ij}} q_{ij} = Pop_i \qquad i \in B \qquad (21)$$

$$c_i = \sum_{j \in N_i} x_{ij} + \sum_{j \in R_{ij}} r_{ij} - 1 \qquad i \in O \qquad (22)$$

$$c_i = \sum_{j \in N_i} x_{ij} + \sum_{j \in R_{ij}} r_{ij} + z_i - 1 \qquad i \in B \qquad (23)$$

Data Description

Problem Set

Using data from 4 states we create 12 problems (Table 1). Four of these problems are from the original Kentucky problem and utilize 1990 census data. The full Kentucky problem contains 120 counties. This provides two problems because we utilize the State level problems of creating house and senate districts (100 and 38 districts, respectively). We created a 13-county subset of Kentucky. The number of districts was proportional to the number of house and senate districts in the full problem.

The data for the South Carolina problem was obtained from Mehrotra, Johnson and Nemhauser (1998). It is also 1990 census data. We create 3 problems from the data. A 6 district problem – this was the problem used by Mehrotra, Johnson and Nemhauser and is the number of federal congressional districts. We also created a 15-district problem and a 38-district problem that are again analogous to the Kentucky House and Senate problems.

The data for Arizona and Wyoming are from the U.S. Census 2000. We have 3 Arizona problems from this data. The 5 district and 13 district problems are analogous to the Kentucky House and Senate problems. The 8 district problem is the federal congressional districting problem for Arizona.

Table 1 Problems

Problem	***Data Set***	***Counties***	***Districts***
1	KY Full	120	38
2	KY Full	120	100
3	KY Subset	13	4
4	KY Subset	13	11
5	AZ	15	5
6	AZ	15	8
7	AZ	15	13
8	WY	23	7
9	WY	23	19
10	SC	46	6
11	SC	46	15
12	SC	46	38

Adjacency Issue

It was necessary to determine for each county, the sets N_j, R_j of one-ring and two-ring adjacent counties. These counties can be considered for membership in a district with county *j*. Initially, all one-ring and two-ring counties were included in these sets. However, when we solved problems with these sets we obtained solutions that were not compact. We increased the requirements for a county to be a member of these sets. A county is a member of the set, N_j, such that if county *i* and county *j* were in the same district then the convex hull of the two counties would not include a large portion of other counties. To determine R_j we solve an all shortest paths problem for county *j* and $\{i \in R_j \mid d(j,i) = 2\}$.

For each problem we developed two data sets. In the first data set, All Adjacency, the sets N_j, R_j contain all one-ring and two-ring counties for county *j* . The second data set, Reduced Adjacency, used the more restrictive set definition and has a reduced set of one-ring and two-ring counties.

RESULTS

Implementation

These models were implemented in AMPL/CPLEX (version 7.5) on a 866-megahertz, Windows platform.

Execution Time

Each problem was run four times:

1-ring model, All adjacency
1-ring model, Reduced adjacency
2-ring model, All Adjacency
2-ring model, Reduced Adjacency

Of the 48 runs, 21 were infeasible. Of these 21 infeasible runs, all but two were found infeasible by AMPL's presolve. These problems are too tightly constrained by the single ring and/or the reduced adjacencies. For all but Problem 12, at least one of the runs found an optimal solution. A summary of the execution times for the successful runs is given in Table 2. Note that the problem name contains the number of districts to the right of the hyphen.

Table 2 Execution Time

Problem Number	Problem	Number of Counties	Adjacency	Model	Execution Time (seconds)
3	KY13-4	13	All	1-ring	0.30
4	KY13-11	13	All	1-ring	0.17
4	KY13-11	13	Reduced	1-ring	0.11
8	WY-7	23	All	1-ring	0.88
9	WY-19	23	All	1-ring	0.65
12	SC-38	46	All	1-ring	160.00
3	KY13-4	13	All	2-ring	2.50
3	KY13-4	13	Reduced	2-ring	1.50
4	KY13-11	13	All	2-ring	3.40
4	KY13-11	13	Reduced	2-ring	1.80
5	AZ-5	15	All	2-ring	0.26
6	AZ-8	15	All	2-ring	5.30
7	AZ-13	15	All	2-ring	3.30
8	WY-7	23	All	2-ring	52.00
9	WY-19	23	All	2-ring	36.00
10	SC-6	46	All	2-ring	1700.00
11	SC-15	46	All	2-ring	13000.00
11	SC-15	46	Reduced	2-ring	610800.00
12	SC-38	46	All	2-ring	Solution not found in 12 hrs.

Examining the execution time, we see that:

- In general, as the number of counties in a state increases the execution time increases. Arizona and the KY-13 problem have close to the same number of counties (15 and 13 respectively).
- For the 1-ring model, problems with a larger number of districts solved faster than the corresponding problem with a smaller number of districts. This does not hold for the 2-ring model – only the WY problem for the 2-ring case.
- When an optimal solution was found for both the data sets (All and Reduced), the Reduced data set had the shorter execution time except for SC-15.
- The 1-ring model solves to optimality more quickly than 2-ring model.

Objective Function Value

Observing Table 3, we see that

- as the number of districts increases the number of required cuts also increases.
- the 1-ring model may require additional cuts compared to the 2-ring model. For example, WY-7, the 1-ring model requires 3 cuts whereas the 2-ring model requires only 2 cuts.
- The Reduced data set may require more cuts than the data set with All adjacencies. The solutions with the Reduced data set are usually more contiguous and compact.

Model Implementation Issues

Without constraint (9), $p_{ij} \geq [MinFrac(Pop_i)]x_{ij}$, the model will find solutions where $x_{ij} = 1$ and $p_{ij} = 0$. That is a part of county *i* is assigned to a district with seat *j* but no population is actually assigned from county *i* to seat *j*. The reason the model will create such a solution (despite the objective function that will keep as many $x_{ij} = 0$ as possible) is to create a link to a county in the 2-ring set. Such a district is not contiguous without an assignment of population. The issue with constraint (9) is that it adds a parameter to the model that must be set, *MinFrac*. The value of this parameter can have a direct effect on the objective function. For instance, in the KY-13 problem, 3 of the 7 feasible problems required an additional cut when this constraint was added. Of the 3 instances, only one actually had the non-contiguity problem in the optimal solution.

Table 3 Objective Function Value

Problem Number	Problem	Adjacency	Model	Objective Function Value
3	KY13-4	All	1-ring	1
3	KY13-4	All	2-ring	1
3	KY13-4	Reduced	2-ring	1
4	KY13-11	All	1-ring	8
4	KY13-11	Reduced	1-ring	9
4	KY13-11	All	2-ring	8
4	KY13-11	Reduced	2-ring	9
5	AZ-5	All	2-ring	2
6	AZ-8	All	2-ring	5
7	AZ-13	All	2-ring	10
8	WY-7	All	1-ring	3
8	WY-7	All	2-ring	2
9	WY-19	All	1-ring	11
9	WY-19	All	2-ring	11
10	SC-6	All	2-ring	1
11	SC-15	All	2-ring	4
11	SC-15	Reduced	2-ring	6
12	SC-38	All	1-ring	24
12	SC-38	All	2-ring	Solution not found in 12 hours

A county in the 3rd ring or further cannot be assigned to a district seat. Is this a problem? Maybe not because we want districts to be geographically compact. As such, it might avoid more problems than it creates. However, it is likely that for some problems the number of cuts could be reduced if a third ring was allowed.

Conclusion

The Kentucky Redistricting problem has been introduced and two mixed integer programming model variations formulated. Though the full Kentucky problem has not yet been successfully solved, smaller problems that were formulated analogously to the Kentucky problem have been successfully solved.

There are definitely trade-offs between the number of cuts required in a districting solution and other desirable characteristics. Solutions with a minimum number of cuts allowing all adjacent counties lead to districts that are not compact. If we improve the compactness by implementing a more constraining definition of adjacency, the optimal number of cuts often increases.

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